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LETTERS TO THE EDITOR

FREE VIBRATIONS OF CIRCULAR PLATES OF VARYING THICKNESS PARTIALLY EMBEDDED IN AN ELASTIC FOUNDATION

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1. Introduction

The present study was originated by the dynamic analysis of the foundation of machinery partially embedded in a Winkler type medium (see Figure 1), and deals with the determination of the lower natural frequencies of transverse vibration corresponding to symmetric and antisymmetric modes. The classical Rayleigh–Ritz method using a polynomial approximation is employed and classical plate theory is applied to describe the dynamic behavior of the plate.

A review of the available literature shows that several cases with circular symmetry have been investigated: the case of stepped annular plates resting on elastic foundations has been studied recently by Wang [1]; the case of axisymmetric vibrations of circular plates with variable thickness (linear as well as parabolic) resting on an elastic foundation of Winkler type has been discussed by Gupta *et al*. [2], for both clamped and simply supported plates; and the case of plates with stepped thickness on elastic foundations have been investigated by Ju *et al*. [3]. On the other hand, the general case of symmetric and antisymmetric modes, for circular plates with partially flat and partially linear thickness, partial Winkler type foundation and general boundary conditions, seems not to have been reported as yet.

2. Solution by means of the Rayleigh–Ritz method

In the case of normal modes of vibration of the vibrating system shown in Figure 1, one takes

$$w(\bar{r}, t) = W(\bar{r})\, e^{i\omega t} \tag{1}$$

for the plate transverse displacement, and then introduces the following approximation, convenient in the case of both axisymmetric and antisymmetric modes of vibration:

$$W(\bar{r}) \simeq W_a(\bar{r}) = \cos k\theta \sum_{j=0}^{J} \sum_{i=0}^{2} A_j C_{ij} R^{\gamma_i + j + k}. \tag{2}$$

$$C_{ij} = \{1, \alpha_j, \beta_j\}, \qquad k = 0, 1, 2, \ldots \qquad \text{and} \qquad \gamma_i = \{0, 2, 4\}, \tag{3}$$

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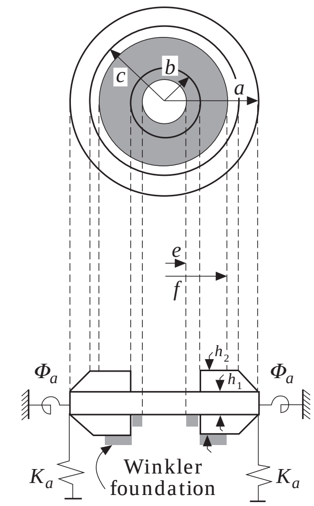

Figure 1. The system under study.

where the α_j's and the β_j's are determined by substituting each co-ordinate function into the governing boundary conditions.

If use is made of the dimensionless variable $r = \bar{r}/a$, these boundary conditions can be written in the form [1]

$$\Phi_a \frac{\partial W}{\partial r}\bigg|_{r=1} = -\left[\frac{\partial^2 W}{\partial r^2} + \mu_1\left(\frac{1}{r}\frac{\partial W}{\partial r} + \frac{1}{r^2}\frac{\partial^2 W}{\partial \theta^2}\right)\right]_{r=1}, \tag{4a}$$

$$K_a W\bigg|_{r=1} = +\left[\frac{\partial(\Delta W)}{\partial r} + \frac{(1-\mu_1)}{r}\frac{\partial}{\partial \theta}\left(+\frac{1}{r}\frac{\partial^2 W}{\partial r \partial \theta} - \frac{1}{r^2}\frac{\partial W}{\partial \theta}\right)\right]_{r=1}, \tag{4b}$$

where Δ is the Laplace operator, Φ_a is the non-dimensional flexibility coefficient of the rotational boundary spring and K_a is the non-dimensional translational spring constant, given by $\Phi_a = a/\phi_a D_1$ and $K_a = k_a a^3/D_1$, and, as usual,

$$D_1 = E_1 h_1^3/12(1-\mu_1^2). \tag{5}$$

Substituting expressions (3) in equations (4) one obtains

$$\sum_{i=0}^{2} C_{ij}((\gamma_i + j + k)[\gamma_i + j + k - 1 + \mu_1(1-k^2)]\Phi_a + 1) = 0, \tag{6a}$$

$$\sum_{i=0}^{2} C_{ij}((\gamma_i + j + k - 2)[(\gamma_i + j + k)^2 k^2] - (1-\mu_1)k^2(\gamma_i + j + k - 1) - K_a) = 0, \tag{6b}$$

where $j = 0, 1, 2, \ldots, J$.

The appropriate functional of the problem is

$$J(W) = U_p + U_b + U_f - T_p, \tag{7}$$

where

$$U_p = \pi D_1 \int_0^1 \left[(\Delta W)^2 + 2(1-\mu_1)\left(\frac{1}{r}\frac{\partial^2 W}{\partial r \partial \theta} - \frac{1}{r^2}\frac{\partial W}{\partial \theta}\right)^2 \right.$$

$$\left. -2(1-\mu_1)\frac{\partial^2 W}{\partial r^2}\left(\frac{1}{r^2}\frac{\partial^2 W}{\partial \theta^2} + \frac{1}{r}\frac{\partial W}{\partial r}\right)^2 \right] r\,\mathrm{d}r$$

$$+\pi D_2 \int_{\eta_1}^{\eta_2} \left[(\Delta W)^2 + 2(1-\mu_2)\left(\frac{1}{r}\frac{\partial^2 W}{\partial r \partial \theta} - \frac{1}{r^2}\frac{\partial W}{\partial \theta}\right)^2 \right.$$

$$\left. -2(1-\mu_2)\frac{\partial^2 W}{\partial r^2}\left(\frac{1}{r^2}\frac{\partial^2 W}{\partial \theta^2} + \frac{1}{r}\frac{\partial W}{\partial r}\right)^2 \right] r\,\mathrm{d}r$$

$$+\pi D_3 \int_{\eta_2}^{1} ((S_a(1-r)+1)^3 - 1)\left[(\Delta W)^2 + 2(1-\mu_2)\left(\frac{1}{r}\frac{\partial^2 W}{\partial r \partial \theta} - \frac{1}{r^2}\frac{\partial W}{\partial \theta}\right)^2 \right.$$

$$\left. -2(1-\mu_2)\frac{\partial^2 W}{\partial r^2}\left(\frac{1}{r^2}\frac{\partial^2 W}{\partial \theta^2} + \frac{1}{r}\frac{\partial W}{\partial r}\right)^2 \right] r\,\mathrm{d}r, \qquad (8a)$$

TABLE 1

The fundamental axisymmetric frequency coefficient $\Omega = (\omega a^2 h_2/h_1)\sqrt{\rho_1 h_1/D_1}$ *for a circular plate with linearly varying thickness; comparison with reference* [5]; $\mu_1 = \mu_2 = 0{\cdot}25$, $K_f = 0$, $a/(\phi_a D_1) = 0$, $\eta_1 = \eta_2 = 0$

		$k_a a^3 h_1^3/D_1 h_2^3$					
h_2/h_1		1	8	32	128	1024	∞
1	Present study	1·3724	3·2111	4·2856	4·7048	4·8414	4·8602
	Reference [2]	1·372	3·211	4·285	4·704	4·840	4·860
1·5	Present study	1·5097	3·0827	3·6878	3·8829	3·9431	3·9516
	Reference [2]	—	—	—	—	—	3·952
5/3	Present study	1·5384	3·0261	3·5470	3·7106	3·7599	3·7666
	Reference [2]	1·538	3·026	3·548	3·710	3·760	3·7667
2·329	Present study	1·6065	2·8190	3·1552	3·2506	3·2751	3·2782
	Reference [2]	—	—	—	—	—	3·277
2·5	Present study	1·6162	2·7740	3·0835	3·1676	3·1886	3·1913
	Reference [2]	1·623	2·785	3·078	3·162	3·187	3·191
5	Present study	1·6347	2·4079	2·5064	2·5267	2·5329	2·5329
	Reference [2]	1·952	2·481	2·550	2·567	2·5372	2·573

$$U_b = \pi D_1 \left[\Phi_a \left(\frac{\partial W}{\partial r} \right)^2_{r=1} + K_a (W^2)_{r=1} \right], \qquad U_f = \pi K_f \int_{\eta_3}^{\eta_4} W^2 r \, \mathrm{d}r \qquad (8b, 9)$$

are the potential energies corresponding to the plate strain, the plate boundary restraints, and the foundation elastic deformation respectively, while

$$T_p = \pi D_1 \Omega^2 \left[\int_0^1 W^2 r \, \mathrm{d}r + (h-1) \frac{\rho_2}{\rho_1} \int_{\eta_1}^{\eta_2} W^2 r \, \mathrm{d}r + S_a \frac{\rho_2}{\rho_1} \int_{\eta_2}^{1} W^2 r (1-r) \, \mathrm{d}r \right] \qquad (10)$$

is the kinetic energy of the plate. Here $\Omega^2 = \rho_1 h_1 a^4 \omega^2 / D_1$, $h = h_2/h_1$, $\eta_1 = b/a$, $\eta_2 = c/a$, $\eta_3 = e/a$ and $\eta_4 = f/a$ are the frequency coefficient, the dimensionless height and radius of step, and the dimensionless radius of inner and outer borders of the foundation respectively. Also $S_a = (h_2 - h_1)/(a - c)$, $K_f = k_f a^4 / D_1$, $D_2 = E_2(h_2^3 - h_1^3)/12(1 - \mu_2^2)$ and $D_3 = E_2 h_1^3 / 12(1 - \mu_2^2)$.

TABLE 2

A comparison of frequencies $\{\Omega h_2/(\sqrt{12}h)\}$ *for the first three symmetric modes and for two different geometrical configurations*

Configuration (a): $K_a = \infty$, $\Phi_a = 0$, $\eta_1 = \eta_3 = 0$, $\eta_2 = \eta_4 = 1$, $\mu_1 = \mu_2 = 0{\cdot}3$, $h_2 = 0{\cdot}1$, $K_f = 0$ (without foundation)

		$\alpha = 1 - h^{-1}$					
Mode		−0·5	−0·3	−0·1	0·1	0·3	0·5
Ω_{00}	Present study	0·1817	0·1659	0·1503	0·1346	0·1188	0·1025
	Reference [2]	0·1817	0·1659	0·1502	0·1346	0·1188	0·1024
	Reference [6]	0·1816	0·1659	0·1503	0·1346	0·1188	0·1025
Ω_{01}	Present study	1·0906	0·9979	0·9050	0·8107	0·7141	0·6136
	Reference [2]	1·1046	1·0047	0·9069	0·8106	0·7143	0·6143
	Reference [6]	1·0895	0·9977	0·9049	0·8105	0·7138	0·6132
Ω_{02}	Present study	2·7115	2·4876	2·2583	2·0283	1·7929	1·5440
	Reference [2]	3·4547	3·0419	2·6392	2·2512	1·8260	1·5587
	Reference [6]	2·6771	2·4686	2·2524	2·0268	1·7919	1·5418

Configuration (b): $K_a = \infty$, $\Phi_a = \infty$, $\eta_1 = \eta_3 = 0$, $\eta_2 = \eta_4 = 1$, $\mu_1 = \mu_2 = 0{\cdot}3$, $h_2 = 0{\cdot}1$, $K_f h_2^3 / 12 a^3 h^3 (1 - \mu^2) = 0.01$ (with foundation)

		$\alpha = 1 - h^{-1}$					
Mode		−0·5	−0·3	−0·1	0·1	0·3	0·5
Ω_{00}	Present study	0·4981	0·4649	0·4351	0·4099	0·3910	0·3802
	Reference [2]	0·4986	0·4649	0·4350	0·4099	0·3910	0·3801
	Reference [6]	0·4983	0·4649	0·4351	0·4099	0·3910	0·3802
Ω_{01}	Present study	1·5051	1·3804	1·2516	1·1224	0·9927	0·8637
	Reference [2]	1·5500	1·4024	1·2601	1·1238	0·9929	0·8657
	Reference [6]	1·5079	1·3804	1·2512	1·1224	0·9926	0·8636
Ω_{02}	Present study	3·3419	3·0230	2·7354	2·4499	2·1607	1·8570
	Reference [2]	4·6432	4·0459	3·4624	2·8998	2·3714	1·9032
	Reference [6]	3·2584	3·0089	2·7284	2·4476	2·1594	1·8546

In accordance with the Ritz method, one requires that

$$\partial J[W_a]/\partial A_j = 0, \qquad j = 0, 1, 2, \ldots, J, \tag{11}$$

and from the non-triviality conditions one obtains the frequency determinant.

TABLE 3

The natural frequency coefficients for symmetric and antisymmetric modes as a function of the foundation constant and geometrical configuration; $K_a = 0$, $\Phi_a = 0$, $\eta_1 = \eta_3 = 0$

			$\eta_2 = \eta_4 = 0{\cdot}25$			$\eta_2 = \eta_4 = 0{\cdot}5$			$\eta_2 = \eta_4 = 0{\cdot}75$		
h	K_f	k	Ω_{k0}	Ω_{k1}	Ω_{k2}	Ω_{k0}	Ω_{k1}	Ω_{k2}	Ω_{k0}	Ω_{k1}	Ω_{k2}
1·0	0	0	0·0000	9·0036	38·473	0·0000	9·0036	38·473	0·0000	9·0036	38·473
		1	0·0000	20·475	59·874	0·0000	20·475	59·874	0·0000	20·475	59·874
		2	5·3583	35·260	84·384	5·3583	35·260	84·384	5·3583	35·260	84·384
	10	0	0·7794	9·1218	38·518	1·5345	9·2908	38·530	2·3264	9·3206	38·573
		1	0·1974	20·488	59·890	0·7862	20·584	59·915	1·7687	20·646	59·937
		2	5·3587	35·262	84·388	5·3774	35·304	84·415	5·5475	35·365	84·427
	100	0	2·1901	10·155	38·929	3·8351	11·900	39·049	6·2002	12·422	39·466
		1	0·6186	20·611	60·038	2·3713	21·549	60·290	5·3254	22·195	60·503
		2	5·3616	35·277	84·425	5·5416	35·690	84·696	6·9671	36·303	84·815
	1000	0	3·7758	16·701	43·297	5·7607	26·490	44·862	12·141	31·813	47·794
		1	1·7977	21·721	61·479	5·4352	29·134	64·250	12·370	35·624	65·961
		2	5·3897	35·420	84·788	6·6762	39·123	87·545	13·117	45·123	88·614
1·5	0	0	0·0000	12·618	50·825	0·0000	13·509	54·657	0·0000	13·937	58·843
		1	0·0000	27·134	78·249	0·0000	29·603	83·192	0·0000	31·442	90·394
		2	7·3441	45·078	108·66	7·8549	48·647	115·94	8·1522	53·108	125·54
	10	0	0·7128	12·617	50·845	1·3796	13·628	54·681	2·0058	14·068	58·885
		1	0·1856	27·141	78·256	0·7224	29·650	83·211	1·5548	31·514	90·421
		2	7·3443	45·078	108·67	7·8657	48·666	115·95	8·2514	53·153	125·56
	100	0	2·1633	13·153	51·028	4·0482	14·746	54·899	6·0745	15·306	59·269
		1	0·5850	27·196	78·317	2·2503	30·076	83·380	4·8526	32·164	90·668
		2	7·3460	45·085	108·68	7·9608	48·833	116·08	9·0855	53·556	125·73
	1000	0	4·9606	17·443	52·914	7·8573	24·788	57·182	13·984	27·519	63·046
		1	1·7938	27·732	78·923	6·2375	34·047	85·114	13·625	38·659	93·111
		2	7·3626	45·146	108·82	8·7811	50·445	117·37	14·527	57·552	127·41
2·0	0	0	0·0000	16·439	62·850	0·0000	18·335	70·679	0·0000	19·082	79·799
		1	0·0000	33·797	96·020	0·0000	39·006	106·48	0·0000	42·859	121·86
		2	9·5799	54·806	131·67	10·637	62·087	146·96	11·120	71·738	167·71
	10	0	0·6581	16·468	62·862	1·2525	18·399	70·692	1·7793	19·153	79·822
		1	0·1756	33·801	96·024	0·6707	39·032	106·49	1·4010	42·898	121·87
		2	9·5800	54·806	131·67	10·644	62·097	146·97	11·181	71·763	167·72
	100	0	2·0442	16·735	62·963	3·8453	18·983	70·811	5·5416	19·808	80·031
		1	0·5545	33·832	96·056	2·1075	39·265	106·59	4·4084	43·249	122·01
		2	9·5811	54·809	131·68	10·705	62·189	147·04	11·708	71·982	167·81
	1000	0	5·4882	19·358	63·998	9·3207	25·154	72·023	14·939	26·954	82·116
		1	1·7286	34·139	96·383	6·2763	41·534	107·57	13·278	46·797	123·37
		2	9·5918	54·843	131·75	11·274	63·088	147·77	15·875	74·176	168·75

3. NUMERICAL RESULTS

In Tables 1 and 2 is depicted a comparison with values available in the literature. Very good agreement is observed. The new results for the case under study are presented in Tables 3–6. Calculations have been made with $\mu_1 = \mu_2 = 0{\cdot}3$ and $J = 5$. Each table corresponds to a different boundary condition, and several geometrical configurations and elastic constants of the foundation have been considered. The first nine

TABLE 4

As Table 3, *but* $K_a = \infty$

			$\eta_2 = \eta_4 = 0{\cdot}25$			$\eta_2 = \eta_4 = 0{\cdot}5$			$\eta_2 = \eta_4 = 0{\cdot}75$		
h	K_f	k	Ω_{k0}	Ω_{k1}	Ω_{k2}	Ω_{k0}	Ω_{k1}	Ω_{k2}	Ω_{k0}	Ω_{k1}	Ω_{k2}
1·0	0	0	4·9352	29·723	74·257	4·9352	29·723	74·257	4·9352	29·723	74·257
		1	13·898	48·485	102·92	13·898	48·485	102·92	13·898	48·485	102·92
		2	25·613	70·121	134·74	25·613	70·121	134·74	25·613	70·121	134·74
	10	0	5·1347	29·781	74·276	5·5334	29·801	74·293	5·8072	29·854	74·304
		1	13·911	48·502	102·94	14·029	48·541	102·95	14·205	48·559	102·96
		2	25·614	70·124	134·74	25·653	70·159	134·75	25·765	70·169	134·76
	100	0	6·6084	30·302	74·445	9·2284	30·516	74·617	10·855	31·007	74·727
		1	14·025	48·648	103·06	15·133	49·047	103·15	16·706	49·221	103·26
		2	25·625	70·156	134·79	26·001	70·501	134·89	27·090	70·604	135·00
	1000	0	12·589	35·529	76·271	21·335	39·032	77·816	30·577	40·844	78·894
		1	15·021	50·058	104·27	22·164	54·318	105·18	32·102	55·484	106·32
		2	25·725	70·463	135·27	28·925	73·910	136·31	37·634	74·876	137·36
1·5	0	0	6·3684	38·057	95·630	6·7797	40·224	101·42	7·1524	43·413	106·75
		1	17·273	62·078	133·30	18·584	65·511	139·95	20·069	69·943	146·88
		2	31·447	88·783	172·42	33·273	94·468	181·85	36·245	100·16	191·75
	10	0	6·4744	38·084	95·639	7·0827	40·259	101·44	7·5775	43·472	106·77
		1	17·279	62·085	133·30	18·646	65·537	139·96	20·211	69·976	146·89
		2	31·448	88·784	172·43	33·291	94·485	181·85	36·315	100·18	191·76
	100	0	7·3464	38·324	95·724	9·3557	40·580	101·58	10·665	43·996	106·96
		1	17·335	62·147	133·36	19·188	65·775	140·05	21·446	70·271	147·03
		2	31·452	88·796	172·45	33·449	94·641	181·92	36·940	100·37	191·87
	1000	0	12·518	40·781	96·596	20·393	44·139	103·04	25·937	48·933	108·84
		1	17·871	62·760	133·93	23·631	68·199	140·98	31·140	73·172	148·42
		2	31·497	88·916	172·65	34·934	96·193	182·63	42·640	102·31	192·97
2·0	0	0	7·7531	46·007	115·79	8·6020	50·007	127·70	9·3923	56·583	137·53
		1	20·382	74·885	161·23	22·947	81·742	174·83	26·179	90·569	189·73
		2	37·035	106·29	208·25	40·468	117·67	226·78	46·608	128·85	247·74
	10	0	7·8187	46·022	115·80	8·7828	50·028	127·71	9·6392	56·616	137·54
		1	20·385	74·888	161·23	22·983	81·757	174·83	26·261	90·588	189·74
		2	37·035	106·29	208·26	40·478	117·68	226·79	46·649	128·86	247·75
	100	0	8·3817	46·160	115·85	10·259	50·213	127·79	11·628	56·913	137·65
		1	20·419	74·922	161·26	23·302	81·896	174·89	26·982	90·754	189·82
		2	37·038	106·30	208·27	40·568	117·77	226·83	47·008	128·97	247·81
	1000	0	12·437	47·564	116·36	19·247	52·167	128·62	23·596	59·804	138·71
		1	20·750	75·258	161·58	26·179	83·296	175·40	33·326	92·403	190·62
		2	37·063	106·36	208·37	41·439	118·65	227·25	50·451	130·08	248·45

eigenvalues are reported for each case. Both the symmetric ($k = 0$) and the antisymmetric ($k = 1$ and $k = 2$) modes are reported. One can conclude that, in view of the good agreement observed and the simplicity, the approximate analytical solution presented herein is quite convenient, simple and accurate for the analysis of this class of systems.

TABLE 5

As Table 4, *but* $\Phi_a = \infty$

			$\eta_2 = \eta_4 = 0{\cdot}25$			$\eta_2 = \eta_4 = 0{\cdot}5$			$\eta_2 = \eta_4 = 0{\cdot}25$		
h	K_f	k	Ω_{k0}	Ω_{k1}	Ω_{k2}	Ω_{k0}	Ω_{k1}	Ω_{k2}	Ω_{k0}	Ω_{k1}	Ω_{k2}
1.0	0	0	10·216	39·771	89·204	10·216	39·771	89·204	10·216	39·771	89·204
		1	21·260	60·847	120·60	21·260	60·847	120·60	21·260	60·847	120·60
		2	34·877	84·589	155·34	34·877	84·589	155·34	34·877	84·589	155·34
	10	0	10·360	39·813	89·218	10·594	39·825	89·234	10·688	39·884	89·246
		1	21·275	60·863	120·61	21·386	60·886	120·62	21·485	60·915	120·63
		2	34·879	84·593	155·35	34·924	84·620	155·35	35·008	84·634	155·36
	100	0	11·546	40·192	89·345	13·507	40·311	89·508	14·251	40·884	89·630
		1	21·410	61·006	120·72	22·472	61·245	120·80	23·407	61·525	120·89
		2	34·894	84·630	155·40	35·345	84·898	155·47	36·165	85·042	155·56
	1000	0	18·327	44·236	90·699	28·254	45·660	92·260	33·021	49·756	93·378
		1	22·613	62·409	121·76	30·561	65·036	122·58	37·519	67·301	123·48
		2	35·044	84·992	155·89	39·028	87·729	156·67	46·117	89·009	157·48
1·5	0	0	11·573	49·459	113·41	11·922	51·745	119·04	12·505	53·830	123·88
		1	25·221	76·562	154·18	26·521	79·965	161·20	27·570	83·705	167·98
		2	41·866	106·07	196·21	43·982	111·90	206·20	46·321	117·04	216·01
	10	0	11·652	49·479	113·42	12·130	51·771	119·05	12·764	53·883	123·90
		1	25·227	76·569	154·18	26·581	79·985	161·21	27·683	83·737	167·99
		2	41·867	106·07	196·21	43·913	111·92	206·20	46·384	117·06	216·02
	100	0	12·333	49·664	113·49	13·856	52·005	119·18	14·888	54·362	124·07
		1	25·290	76·631	154·23	27·109	80·166	161·29	28·681	84·018	168·12
		1	41·873	106·08	196·23	44·097	112·05	206·26	46·949	117·24	216·11
	1000	0	17·323	51·579	114·18	24·870	54·463	120·50	28·432	58·925	125·80
		1	25·889	77·241	154·77	31·728	82·013	162·15	37·205	86·775	169·32
		2	41·935	106·22	196·44	45·849	113·41	206·87	52·249	119·07	217·04
2·0	0	0	12·930	58·461	135·84	13·646	62·674	147·36	14·557	67·576	156·69
		1	28·800	91·094	185·60	31·413	97·833	199·59	33·738	106·04	214·23
		2	48·344	125·85	235·87	52·134	137·57	255·28	57·380	148·07	275·47
	10	0	12·980	58·473	135·85	13·779	62·689	147·37	14·723	67·607	156·70
		1	28·804	91·098	185·60	31·448	97·845	199·59	33·806	106·06	214·24
		2	48·344	125·85	235·87	52·146	137·58	255·29	57·418	148·08	275·48
	100	0	13·424	58·584	135·89	14·917	62·829	147·45	16·141	67·886	156·80
		1	28·840	91·131	185·63	31·758	97·956	199·64	34·416	106·22	214·31
		2	48·348	125·85	235·88	52·248	137·66	255·32	57·751	148·18	275·53
	1000	0	17·076	59·712	136·31	23·253	64·271	148·21	26·420	70·608	157·79
		1	29·196	91·467	185·94	34·646	99·072	200·12	39·994	107·81	215·00
		2	48·381	125·93	236·00	53·243	138·45	255·68	60·977	149·22	276·08

TABLE 6

As Table 5, but $K_a = 32$ and $\Phi_a = 8$

			$\eta_2 = \eta 4 = 0{\cdot}25$			$\eta_2 = \eta 4 = 0{\cdot}5$			$\eta_2 = \eta 4 = 0{\cdot}75$		
h	K_f	k	Ω_{k0}	Ω_{k1}	Ω_{k2}	Ω_{k0}	Ω_{k1}	Ω_{k2}	Ω_{k0}	Ω_{k1}	Ω_{k2}
1·0	0	0	6·1693	16·660	46·336	6·1693	16·660	46·336	6·1693	16·660	46·336
		1	9·5080	27·836	68·215	9·5080	27·836	68·215	9·5080	27·836	68·215
		2	13·495	42·629	93·019	13·495	42·629	93·019	13·495	42·629	93·019
	10	0	6·2882	16·723	46·369	6·5455	16·777	46·380	6·7825	16·795	46·415
		1	9·5141	27·849	68·230	9·5837	27·922	68·247	9·7674	27·945	68·773
		2	13·496	42·631	93·024	13·512	42·672	93·046	13·616	42·711	93·060
	100	0	7·1812	17·300	46·672	8·9150	17·964	46·782	10·617	18·079	47·133
		1	9·5678	27·969	68·369	10·196	28·696	68·544	11·753	28·950	68·733
		2	13·499	42·648	93·062	13·600	43·058	93·284	14·624	43·450	93·422
	1000	0	10·112	22·658	49·995	13·307	30·849	51·001	20·037	33·247	54·166
		1	10·015	29·080	69·740	13·142	35·729	71·669	20·107	39·434	73·688
		2	13·530	42·819	93·442	14·781	46·609	95·727	20·796	50·829	96·984
1·5	0	0	5·9945	19·465	58·661	5·9194	19·982	62·424	5·8183	20·174	65·172
		1	9·4476	34·170	86·845	9·3806	36·199	91·461	9·2061	37·275	97·414
		2	14·626	52·489	117·83	14·790	55·832	124·78	14·713	59·157	133·35
	10	0	6·0678	19·499	58·677	6·1554	20·051	62·445	6·2274	20·253	65·210
		1	9·4512	34·176	86·852	9·4250	36·240	91·477	9·3713	37·332	97·440
		2	14·626	52·489	117·84	14·779	55·851	124·79	14·783	59·196	133·37
	100	0	6·6658	19·811	58·825	7·8586	20·699	62·630	9·0386	20·989	65·553
		1	9·4828	34·231	86·910	9·8060	36·612	91·620	10·720	37·845	97·673
		2	14·628	52·496	117·85	14·880	56·020	124·91	15·388	59·550	133·53
	1000	0	9·7005	22·985	60·355	13·336	28·335	64·523	19·036	29·678	68·957
		1	9·7759	34·758	87·496	12·419	40·217	93·085	18·433	43·132	99·983
		2	14·644	52·566	118·00	15·609	57·656	126·07	20·077	63·094	135·16
2·0	0	0	5·7985	22·551	70·768	5·6531	23·788	78·594	5·4586	24·032	85·085
		1	9·2950	40·472	104·66	9·1263	44·991	114·84	8·8193	47·716	128·67
		2	15·990	62·087	141·04	16·414	68·958	155·89	16·292	76·919	175·57
	10	0	5·8520	22·573	70·778	5·8280	23·833	78·606	5·7760	24·085	85·107
		1	9·2976	40·475	104·67	9·1590	45·015	114·85	8·9458	47·750	128·69
		2	15·990	62·087	141·04	16·420	68·969	155·90	16·339	76·942	175·58
	100	0	6·3030	22·769	70·865	7·1675	24·245	78·711	8·0557	24·565	85·303
		1	9·3206	40·506	104·70	9·4452	45·229	114·93	10·004	48·055	128·82
		2	15·991	62·091	141·05	16·473	69·061	155·97	16·775	77·145	175·68
	1000	0	9·1232	24·777	71·750	13·008	28·944	79·782	17·894	29·882	87·262
		1	9·5419	40·809	105·01	11·699	47·342	115·80	16·869	51·157	130·13
		2	16·001	62·128	141·12	16·982	69·974	156·65	20·321	79·182	176·59

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